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Translated by M.D.F.

ON THE EFFECT OF AN AXISYMMETRIC NORMAL LOADING ON AN ELASTIC SPHERE

PMM Vol. 33, №6, 1969, pp. 1029-1033
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(Received April 4, 1969)

A solution of the problem of deformation of a sphere under normal loadings is obtained by quadratures. The Green's function of the boundary value problem is written out in finite form. In contrast to an analogous series solution [1], the solution found admits of nonsmooth loadings. As an example, the problem of compression of a sphere by concentrated forces is solved in closed form; the solution is expressed in terms of a hypergeometric function.

It is known from [1] that the solution of the equilibrium equations of an elastic body in displacements

$$\frac{2(1-\nu)}{1-2\nu}\operatorname{grad}\operatorname{div}\mathbf{u}-\operatorname{rot}\operatorname{rot}\mathbf{u}=0$$

with the boundary conditions

$$\tau_{r\theta}=0,\quad \sigma_r=\sigma(\theta),\quad \tau_{\varphi r}=0\qquad \text{for } r=R$$

in a spherical coordinate system r, θ, φ has the following form:

$$u_r=\frac{R}{4\pi G}\int_0^\pi\sigma(\alpha)\sin\alpha\, d\alpha\int_0^{\frac{1}{2}\pi}d\psi\left\{\sum_{n=2}^{\infty}P_n(\lambda)\left[A_{1n}\left(\frac{r}{R}\right)^{n+1}+\right.\right.$$
$$\left.\left.+A_{2n}\left(\frac{r}{R}\right)^{n-1}\right]+\frac{2(1-2\nu)}{1+\nu}\frac{r}{R}\right\}$$

$$u_\theta=\frac{R}{4\pi G}\int_0^\pi\sigma(\alpha)\sin\alpha\, d\alpha\frac{\partial}{\partial\theta}\int_0^{\frac{1}{2}\pi}d\psi\sum_{n=2}^{\infty}P_n(\lambda)\left[A_{3n}\left(\frac{r}{R}\right)^{n+1}+A_{4n}\left(\frac{r}{R}\right)^{n-1}\right]\qquad(1)$$

$$\lambda=\cos(\theta+\alpha)+2\sin\theta\sin\alpha\sin^2\psi$$

Here $P_n(\lambda)$ are Legendre polynomials, and the coefficients A_{in} are rational fraction functions of n

$$A_{1n} = -\frac{(2n+1)(n-2+4\nu)(n+1)}{n^2+(1+2\nu)n+1+\nu}, \qquad A_{2n} = \frac{(2n+1)(n^2+2n-1+2\nu)n}{(n-1)[n^2+(1+2\nu)n+1+\nu]}$$

$$A_{3n} = -\frac{(2n+1)(n+5-4\nu)}{n^2+(1+2\nu)n+1+\nu}, \qquad A_{4n} = \frac{(2n+1)(n^2+2n-1+2\nu)}{(n-1)[n^2+(1+2\nu)n+1+\nu]}$$

Let us prove that the solution of (1) can be represented by quadratures. To do this we expand the A_{in} in elementary fractions

$$A_{1n} = -(2n+1)+4(1-\nu)+\frac{P}{n-n_1}+\frac{\bar{P}}{n+\bar{n}_1}$$

$$A_{2n} = (2n+1)+4(1-\nu)+\frac{2}{n-1}+\frac{Q}{n-n_1}+\frac{\bar{Q}}{n-\bar{n}_1}$$

$$A_{3n} = -2+\frac{S}{n-n_1}+\frac{\bar{S}}{n-\bar{n}_1}$$

$$A_{4n} = 2+\frac{2}{n-1}+\frac{T}{n-n_1}+\frac{\bar{T}}{n-\bar{n}_1}$$

Here n_1 and $\bar{n}_1$ are complex conjugate roots of the equation

$$n^2+(1+2\nu)n+1+\nu=0$$

The constants P, Q, S, T depend only on the Poisson's ratio ν and are given by the formulas

$$\begin{aligned}
P &= 4\nu^2-6\nu+2+i\frac{8\nu^3-12\nu^2+\nu+3}{\sqrt{3-4\nu^2}} \\
Q &= 4\nu^2-2\nu-1+i\frac{8\nu^3-4\nu^2-5\nu+2}{\sqrt{3-4\nu^2}} \\
S &= \frac{12\nu-9}{2}+i\frac{24\nu^2-6\nu-3}{2\sqrt{3-4\nu^2}} \\
T &= \frac{3-4\nu}{2}+i\frac{1+7\nu-6\nu^2}{2\sqrt{3-4\nu^2}}
\end{aligned} \tag{2}$$

It is now evident that the question of the possibility of representing the solution in quadratures reduces to the question of the possibility of a finite representation of the series

$$\sum x^n P_n(\lambda), \qquad \sum n x^n P_n(\lambda), \qquad \sum \frac{x^n}{n-a} P_n(\lambda)$$

The value of the first sum is known [2]

$$\sum_{n=0}^{\infty} x^n P_n(\lambda) = \frac{1}{s}, \qquad s=\sqrt{x^2-2x\lambda+1}$$

The last two series can be summed, as is seen from the following chain of formulas

$$\sum_{n=0}^{\infty} n x^n P_n(\lambda) = x\frac{\partial}{\partial x}\sum_{n=0}^{\infty} x^n P_n(\lambda) = \frac{x(x-\lambda)}{s^3}$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n-a} P_n(\lambda) = x^a \int_0^x dx \sum_{n=0}^{\infty} x^{n-a-1} P_n(\lambda) = \int_0^1 \frac{dy}{y^{1+a}\sqrt{x^2y^2-2xy\lambda+1}}$$

Let us note that the last integral is expressed in terms of elementary functions for

$a=1$, and for $a=n_1$ and $a=\bar{n}_1$ agrees to $1/a$ accuracy with the hypergeometric function of two variables $F_1(-a, {}^1/_2, {}^1/_2, 1-a;\ xe^{i \arccos\lambda}, xe^{-i\arccos\lambda})$. This follows from the integral representation [2]

$$F_1(\alpha, \beta, \beta', \alpha+\mu; u, v) =$$
$$= \frac{1}{B(\mu, \alpha)}\int_0^1 y^{\alpha-1}(1-y)^{\mu-1}(1-uy)^{-\beta}(1-vy)^{-\beta'}\,dy$$

Moreover, the series we are interested in can be integrated with respect to ψ. Utilizing the expression for the sum of the series, it is easy to verify, say, that

$$U(x)\equiv U(x, \theta, \alpha) = \int_0^{{}^1/_2\pi} d\psi \sum_{n=2}^{\infty} x^n P_n(\lambda) = \frac{K(k)}{h} - \frac{\pi}{2}(1+x\cos\theta\cos\alpha)$$
$$h^2 = (1-x)^2 + 4x\sin^2\frac{\theta+\alpha}{2}, \qquad k^2 = \frac{4x\sin\theta\sin\alpha}{h^2} \tag{3}$$

Here $K(k)$ is the complete elliptic integral of the first kind.

Finally, let us represent the solution (1) in the following finite form:

$$u_r(r, \theta) = \frac{R}{2\pi G}\int_0^{\pi}\sigma(\alpha) H_r(r/R, \theta, \alpha)\sin\alpha\, d\alpha \tag{4}$$
$$u_\theta(r, \theta) = \frac{R}{2\pi G}\int_0^{\pi}\sigma(\alpha) H_\theta(r/R, \theta, \alpha)\sin\alpha\, d\alpha$$

where

$$H_r(x, \theta, \alpha) = \frac{1-2\nu}{1+\nu}\frac{\pi x}{2} + \frac{1-x^2}{2x}\left(2x\frac{\partial U}{\partial x} + U\right) + 2(1-\nu)\frac{1+x^2}{x}U + \Big]$$
$$+\frac{1}{x}\operatorname{Re}\int_0^1\left(\frac{Px^2+Q}{y^{1+n_1}} + \frac{1}{y^2}\right)U(xy)\,dy \tag{5}$$

$$H_\theta(x, \theta, \alpha) = \frac{1}{x}\frac{\partial}{\partial\theta}\left[(1-x^2)U + \operatorname{Re}\int_0^1\left(\frac{Sx^2+T}{y^{1+n_1}} + \frac{1}{y^2}\right)U(xy)\,dy\right]$$

Here $U=U(x)$ is defined by (3), the constants P, Q, S, T are given, as before, by (2), and
$$2n_1 = -(1+2\nu) + i\sqrt{3-4\nu^2}$$

Let us investigate the solution obtained.

Proceeding from the representation (5), and utilizing the properties of elliptic integrals, it can be shown that the functions H_r and H_θ are continuous everywhere for $0\leqslant x<1$. They have singularities at the point $\alpha=\theta$ on the surface of the sphere $x=1$ such that to the accuracy of continuous functions

$$\sin\alpha H_r(1, \theta, \alpha) = -2(1-\nu)\ln|\theta-\alpha| + O(1)$$
$$\sin\alpha H_\theta(1, \theta, \alpha) = -{}^1/_2(1-2\nu)\pi\ \operatorname{sign}(\theta-\alpha) + O(1)$$
$$\text{for}\quad \alpha\to\theta$$

Let us write relations (5) explicitly for $\alpha=0$

$$H_r(x, \theta, 0)\frac{2}{\pi} = \frac{1-2\nu}{1+\nu}x + \frac{1-x^2}{2x}\left(\frac{1-x^2}{a^3} - 1 - 3x\cos\theta\right) + \frac{1-a}{x} +$$

$$+2(1-\nu)\frac{1+x^2}{x}\left(\frac{1}{a}-1-x\cos\theta\right)-\cos\theta\left(1+\ln\frac{a+1-x\cos\theta}{2}\right)+$$

$$+\frac{1}{x}\operatorname{Re}(Px^2+Q)\left[\frac{1-F_1(-n_1,{}^1/_2,{}^1/_2,1-n_1;\,xe^{i\theta},\,xe^{-i\theta})}{n_1}+\frac{x\cos\theta}{n_1-1}\right] \quad (6)$$

$$H_\theta(x,\theta,0)\frac{2}{\pi\sin\theta}=(1-x^2)\left(1-\frac{1}{a^3}\right)+\frac{a^2-2ax\cos\theta-1}{a(a+1-x\cos\theta)}+$$

$$+\ln\frac{a+1-x\cos\theta}{2}-\operatorname{Re}\frac{(Sx^2+T)\,[F_1(1-n_1,{}^3/_2,{}^3/_2,2-n_1;\,xe^{i\theta},\,xe^{-i\theta})-1]}{1-n_1}$$

$$a=\sqrt{x^2-2x\cos\theta+1}$$

Let us also note some other properties of the functions H_r and H_θ

$$H_r(x,\theta,\alpha)=H_r(x,\alpha,\theta),\ H_r(x,\theta,\alpha)=H_r(x,\pi-\theta,\pi-\alpha)$$

$$H_\theta(x,0,\alpha)=H_\theta(x,\pi,\alpha)=0,\quad H_\theta(x,\theta,\alpha)=-H_\theta(x,\pi-\theta,\pi-\alpha)$$

As examples, let us consider the problem of deformation of a weighted sphere equilibrated by a concentrated force, and the problem of compression of a sphere by two concentrated forces applied at its poles. These problems have recently been considered by several authors [1, 3, 4]. The singular part of the solution was isolated by some method or other in these papers, and was represented analytically, and hence the smooth part of the solution remained written in series form.

Let us give closed solutions of the formulated problems.

Example 1. First we turn to the problem of deformation of a weighted sphere of density ρ. The sphere is equilibrated by a concentrated force $F=4/3\pi R^3\rho g$ applied at the pole $\theta=0$, $r=R$. In this case the boundary conditions can be written as follows:

$$\sigma(\theta)\sin\theta=-\frac{F}{2\pi R^2}\delta(\theta),\qquad \tau(\theta)=0$$

where $\delta(\theta)$ is the Dirac delta function.

The corresponding displacements are

$$u_r^\circ(r,\theta)=-\left[\cos\theta\frac{3r^2}{4R^2}\frac{1-2\nu}{1+\nu}+\frac{1}{\pi}H_r(r/R,\theta,0)\right]\frac{F}{4\pi GR} \quad (7)$$

$$u_\theta^\circ(r,\theta)=-\left[\sin\theta\frac{3r^2}{4R^2}\frac{1-2\nu}{1+\nu}+\frac{1}{\pi}H_\theta(r/R,\theta,0)\right]\frac{F}{4\pi GR}$$

As before, the functions H_r and H_θ are given by formulas (6).

The solution (7) is continuous everywhere except at the point of application of the force $\theta=0$, $r=R$. The nature of the discontinuity is determined by the relationships

$$H_r(1,\theta,0)\frac{1}{\pi}=(1-\nu)\left(\frac{1}{\sin\frac{\theta}{2}}-4\cos^2\frac{\theta}{2}\right)+\frac{1-2\nu}{2(1+\nu)}-$$

$$-(1-2\nu)^2\left[\cos\theta\ln\sin\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\right)+2\sin\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\right)\right]+$$

$$+\operatorname{Re}\frac{P+Q}{2}\int_0^1\frac{y^{1-n_1}-1}{y^2}\left[\frac{1}{\sqrt{y^2-2y\cos\theta+1}}-1-y\cos\theta\right]dy \quad (8)$$

$$H_\theta(1,\theta,0)\frac{1}{\pi}=-\frac{1-2\nu}{2}\left[\operatorname{ctg}\frac{\theta}{2}\,\frac{4\sin^2{}^1/_2\theta-4\sin{}^1/_2\theta\cos\theta-1}{1+\sin{}^1/_2\theta}+2\sin\theta\ln\sin\frac{\theta}{2}\times\right.$$

$$\times\left(1+\sin\frac{\theta}{2}\right)\Big]+\frac{\partial}{\partial\theta}\operatorname{Re}\frac{S+T}{2}\int_0^1\frac{y^{1-n_1}-1}{y^2}\left[\frac{1}{\sqrt{y^2-2y\cos\theta+1}}-1-y\cos\theta\right]dy$$

It is easy to see that the integrands are bounded everywhere for $0 \leqslant \theta \leqslant \pi$, $0 \leqslant y \leqslant 1$. The first two members in the relationships (8) agree asymptotically, for $\theta \to 0$, with the isolated singularity in the solutions in [1, 3].

Example 2. Evidently the solution of the problem of compression of a sphere by concentrated forces applied to its poles $r = R$, $\theta = 0$ and $\theta = \pi$ is the superposition of two solutions of type (7), namely

$$u_r(r,\theta) = u_r^*(r,\theta) + u_r^*(r,\pi-\theta)$$
$$u_\theta(r,\theta) = u_\theta^*(r,\theta) - u_\theta^*(r,\pi-\theta)$$

Thus the solution of the problem of deformation of a sphere by an axisymmetric normal loading is represented by the quadratures of (4),(5). The advantage of this representation will be that it is valid even for loadings having a strong discontinuity of a concentrated force type.

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Translated by M. D. F.

CERTAIN TYPE OF INTEGRAL EQUATIONS APPEARING IN CONTACT PROBLEMS OF THE THEORY OF ELASTICITY

PMM Vol. 33, №6, 1969, pp.1034-1041
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(Received May 29, 1969)

A method of investigating the integral equation for the case when its kernel is a meromorphic function with simple poles and double zeros, is presented. The integral equation is reduced to an infinite system of linear algebraic equations which normally has a solution, and this solution is constructed together with that of a certain finite system. A general form of sufficient conditions which must be imposed on the right side of the equation to ensure that it has a unique solution, is derived.

Mixed problems of the theory of elasticity on determination of stresses generated under a die impressed into an elastic layer lying without friction on a rigid foundation [1], and the problem concerning the stresses generated under a wheel with a tyre, fitted on an elastic shaft [2], both lead to an integral equation of the form